

FROBENIUS CONDITION ON A PRETRIANGULATED CATEGORY, AND TRIANGULATION ON THE ASSOCIATED STABLE CATEGORY

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ABSTRACT. As shown by Happel, from any Frobenius exact category, we can construct a triangulated category as a stable category. On the other hand, it was shown by Iyama and Yoshino that if a pair of subcategories $\mathcal{D} \subseteq \mathcal{Z}$ in a triangulated category satisfies certain conditions (i.e., $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair), then \mathcal{Z}/\mathcal{D} becomes a triangulated category. In this article, we consider a simultaneous generalization of these two constructions.

1. INTRODUCTION AND PRELIMINARIES

Throughout this article, we fix an additive category \mathcal{C} . Any subcategory of \mathcal{C} will be assumed to be full, additive and replete. A subcategory is called *replete* if it is closed under isomorphisms.

When we say \mathcal{Z} is an exact category, we only consider an extension-closed subcategory of an abelian category.

For any category \mathcal{K} , we write abbreviately $K \in \mathcal{K}$, to indicate that K is an object of \mathcal{K} . For any $K, L \in \mathcal{K}$, let $\mathcal{K}(K, L)$ denote the set of morphisms from K to L . If \mathcal{M}, \mathcal{N} are full subcategories of \mathcal{K} , then $\mathcal{K}(\mathcal{M}, \mathcal{N}) = 0$ means that $\mathcal{K}(M, N) = 0$ for any $M \in \mathcal{M}$ and $N \in \mathcal{N}$. Similarly, $\mathcal{K}(K, \mathcal{N}) = 0$ means $\mathcal{K}(K, N) = 0$ for any $N \in \mathcal{N}$.

If \mathcal{K} is an additive category and \mathcal{L} is a full additive replete subcategory which is closed under finite direct summands, then \mathcal{K}/\mathcal{L} denotes the quotient category of \mathcal{K} by the ideal generated by \mathcal{L} . The image of $f \in \mathcal{K}(X, Y)$ will be denoted by $\underline{f} \in \mathcal{K}/\mathcal{L}(X, Y)$.

As shown by Happel [H], If we are given a Frobenius exact category \mathcal{E} , then the stable category \mathcal{E}/\mathcal{I} , where \mathcal{I} is the full subcategory of injectives, carries a structure of a triangulated category.

On the other hand, it was shown by Iyama and Yoshino that if $\mathcal{D} \subseteq \mathcal{Z}$ is a pair of subcategories in a triangulated category \mathcal{C} such that $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair, then the quotient category \mathcal{Z}/\mathcal{D} becomes a triangulated category. By definition, $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair if it satisfies

- (1) $\mathcal{C}(\mathcal{Z}, \mathcal{D}[1]) = \mathcal{C}(\mathcal{D}, \mathcal{Z}[1]) = 0$,
- (2) For any object $X \in \mathcal{Z}$, there exists a distinguished triangle

$$X \rightarrow D \rightarrow Z \rightarrow \Sigma X$$

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with $D \in \mathcal{D}$ and $Z \in \mathcal{Z}$,

- (3) For any object $Z \in \mathcal{Z}$, there exists a distinguished triangle

$$X \rightarrow D \rightarrow Z \rightarrow \Sigma X$$

with $X \in \mathcal{Z}$ and $D \in \mathcal{D}$.

In this article, we make a simultaneous generalization of these two constructions, by using a slight modification of a *pretriangulated category* in [BR]. To emphasize this modification, we call it a ‘pseudo-’triangulated category. As in Definition 3.3, a pseudo-triangulated category is an additive category \mathcal{C} with a *pseudo-triangulation* $(\Sigma, \Omega, \triangleright, \triangleleft, \psi)$.

As in Example 4.5, a pseudo-triangulated category \mathcal{C} is abelian if and only if $\Sigma = \Omega = 0$, and \mathcal{C} is triangulated if and only if $\Sigma \cong \Omega^{-1}$. An *extension* in \mathcal{C} is a simultaneous generalization of a short exact sequence in the abelian case, and a distinguished triangle in the triangulated case (Definition 4.1). For an extension-closed subcategory $\mathcal{Z} \subseteq \mathcal{C}$, we define the *Frobenius condition* on it (Definition 5.9). This is equivalent to the ordinary Frobenius condition in the case of $\Sigma = \Omega = 0$, and related to the existence of a mutation pair in the triangulated case (Example 5.10 and Corollary 5.16). As a main theorem, in Theorem 6.17, we show if \mathcal{Z} is Frobenius, then the associated stable category becomes a triangulated category. In the above two cases, this recovers the Happel’s and Iyama-Yoshino’s constructions, respectively.

	$\Sigma = \Omega = 0$	$\Sigma \cong \Omega^{-1}$
Pretriangulated	abelian	triangulated
Extension	short exact sequence	distinguished triangle
Frobenius condition	Frobenius condition	Corollary 5.16
Theorem 6.17	Happel’s construction	Iyama-Yoshino’s construction

2. ONE-SIDED TRIANGULATED CATEGORIES

Definition 2.1 (right triangulation cf. [BM], [BR]). Let $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ be an additive endofunctor, and let $\mathcal{RT}(\mathcal{C}, \Sigma)$ be the category of diagrams of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.$$

A morphism from $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ to $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$ is a triplet (a, b, c) of morphisms $a \in \mathcal{C}(A, A')$, $b \in \mathcal{C}(B, B')$ and $c \in \mathcal{C}(C, C')$, satisfying

$$b \circ f = f' \circ a, \quad c \circ g = g' \circ b, \quad \Sigma a \circ h = h' \circ c.$$

A pair (Σ, \triangleright) of Σ and a full replete subcategory $\triangleright \subseteq \mathcal{RT}(\mathcal{C}, \Sigma)$ is called a *right triangulation* on \mathcal{C} if it satisfies the following conditions. Remark that Σ is not necessarily an equivalence.

- (RTR1) For any $A \in \mathcal{C}$, $0 \rightarrow A \xrightarrow{\text{id}_A} A \rightarrow \Sigma 0 = 0$ is in \triangleright . For any morphism $f \in \mathcal{C}(A, B)$, there exists an object $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in \triangleright .
- (RTR2) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is in \triangleright , then $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$ is also in \triangleright .
- (RTR3) If we are given two objects $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$ in \triangleright and two morphisms $a \in \mathcal{C}(A, A')$ and $b \in \mathcal{C}(B, B')$ satisfying $b \circ f = f' \circ a$, then there exists $c \in \mathcal{C}(C, C')$ such that (a, b, c) is a morphism in \triangleright .

(RTR4) Let

$$\begin{aligned} A &\xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A, \\ A &\xrightarrow{\ell} M \xrightarrow{m} B' \xrightarrow{n} \Sigma A, \\ A' &\xrightarrow{\ell'} M \xrightarrow{m'} B \xrightarrow{n'} \Sigma A' \end{aligned}$$

be objects in \triangleright , satisfying $m' \circ \ell = f$.

Then there exist $g' \in \mathcal{C}(B', C)$ and $h' \in \mathcal{C}(C, \Sigma A')$ such that

$$\begin{aligned} h' \circ g &= n' \quad , \quad h \circ g' = n, \\ g' \circ m &= g \circ m' \quad , \quad (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0, \end{aligned}$$

and

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

is an object in \triangleright . Here we put $f' = m \circ \ell'$.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{n'} & \Sigma A' & & \\ & \searrow \ell & \nearrow m' & \searrow g & \nearrow h' & \searrow \Sigma \ell' & \\ & & M & & C & & \Sigma M \\ & \nearrow \ell' & \searrow m & \nearrow g' & \searrow h & \nearrow -\Sigma \ell & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{n} & \Sigma A & & \end{array}$$

If (Σ, \triangleright) is a right triangulation on \mathcal{C} , we call $(\mathcal{C}, \Sigma, \triangleright)$ a *right triangulated category*.

Caution 2.2. Conditions (RTR4) is slightly different from that in [BM].

Definition 2.3 (left triangulation). Let $\Omega: \mathcal{C} \rightarrow \mathcal{C}$ be an additive endofunctor, and let $\mathcal{LT}(\mathcal{C}, \Omega)$ be the category of diagrams of the form

$$\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C.$$

A morphism in $\mathcal{LT}(\mathcal{C}, \Omega)$ is defined similarly as in Definition 2.1. A pair (Ω, \triangleleft) satisfying conditions (LTR1), (LTR2), (LTR3) and (LTR4) which are dual to (RTR1), (RTR2), (RTR3) and (RTR4) respectively, is called a *left triangulation* on \mathcal{C} , and $(\mathcal{C}, \Omega, \triangleleft)$ is called a *left triangulated category*.

Similarly to the triangulated case, the following are satisfied.

Proposition 2.4. *Let \mathcal{C} be an additive category.*

- (1) *If (Σ, \triangleright) is a right triangulation on \mathcal{C} , then for any object $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ in \triangleright and for any $E \in \mathcal{C}$, the induced sequence*

$$\mathcal{C}(A, E) \leftarrow \mathcal{C}(B, E) \leftarrow \mathcal{C}(C, E) \leftarrow \mathcal{C}(\Sigma A, E) \leftarrow \mathcal{C}(\Sigma B, E) \leftarrow \cdots$$

is exact.

- (2) *Dually for a left triangulation.*

Proof. Left to the reader. □

3. PSEUDO-TRIANGULATED CATEGORY

In this section, we introduce a notion unifying triangulated categories and abelian categories. We make a slight modification of the pretriangulated category in [BR], for the sake of Example 4.5. We call it a ‘pseudo-’triangulated category, to make the reader beware of this modification. Roughly speaking, a pseudo-triangulated category is an additive category endowed with right and left triangulated triangulations, satisfying some gluing conditions (Definition 3.3).

Definition 3.1. Let (Σ, \triangleright) be a right triangulation on \mathcal{C} , and let $f: A \rightarrow B$ be any morphism in \mathcal{C} .

- (1) f is Σ -null if it factors through some object in $\Sigma\mathcal{C}$.
- (2) f is Σ -epic if for any $B' \in \mathcal{C}$ and any $b \in \mathcal{C}(B, B')$, $b \circ f = 0$ implies b is Σ -null.

For a left triangulation (Ω, \triangleleft) , dually we define Ω -null morphisms and Ω -monic morphisms.

Remark 3.2. For any morphism $f \in \mathcal{C}(A, B)$, the following are equivalent.

- (1) f is Σ -epic.
- (2) There exists an object in \triangleright

$$A \xrightarrow{f} B \xrightarrow{g} C \rightarrow \Sigma A$$

such that g is Σ -null.

- (3) For any object in \triangleright

$$A \xrightarrow{f} B \xrightarrow{g} C \rightarrow \Sigma A,$$

g becomes Σ -null.

Dually for Ω -monics.

Definition 3.3. A pseudo-triangulation $(\Sigma, \Omega, \triangleright, \triangleleft, \psi)$ on \mathcal{C} is a pair (Σ, \triangleright) and (Ω, \triangleleft) of right and left triangulations, together with an adjoint natural isomorphism

$$\psi_{A,B}: \mathcal{C}(\Omega A, B) \xrightarrow{\cong} \mathcal{C}(A, \Sigma B) \quad (A, B \in \mathcal{C}),$$

which satisfies the following gluing conditions (G1) and (G2).

- (G1) If $g \in \mathcal{C}(B, C)$ is Σ -epic, then for any objects

$$\begin{aligned} \Omega C &\xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C && \in \triangleleft, \\ A &\xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A && \in \triangleright, \end{aligned}$$

there exists an isomorphism $c \in \mathcal{C}(C', C)$ such that

$$c \circ g' = g \quad \text{and} \quad -\psi(e) \circ c = h'.$$

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A \\ & & \parallel & & \parallel & & \downarrow \cong & \downarrow \exists c & \nearrow \\ \Omega C & \xrightarrow{e} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & & \\ & & \parallel & & \parallel & & \downarrow & & \\ & & \circ & & \circ & & \downarrow & & \downarrow -\psi(e) \end{array}$$

Roughly speaking, this means that any Σ -epic morphism agrees with the ‘cokernel’ of its ‘kernel’.

(G2) Dually, if $f \in \mathcal{C}(A, B)$ is Ω -monic, then for any objects

$$\begin{aligned} A &\xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A && \in \triangleright, \\ \Omega C &\xrightarrow{e'} A' \xrightarrow{f'} B \xrightarrow{g} C && \in \triangleleft, \end{aligned}$$

there exists an isomorphism $a \in \mathcal{C}(A, A')$ such that

$$f' \circ a = f \quad \text{and} \quad -a \circ \psi^{-1}(h) = e'.$$

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C \xrightarrow{h} \Sigma A \\ & \nearrow -\psi^{-1}(h) & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \Omega C & \xrightarrow{e'} & A' & \xrightarrow{f'} & B & \xrightarrow{g} & C \end{array}$$

If we are given a pseudo-triangulation $(\Sigma, \Omega, \triangleright, \triangleleft, \psi)$ on \mathcal{C} , then we call the 6-tuple $(\mathcal{C}, \Sigma, \Omega, \triangleright, \triangleleft, \psi)$ a *pseudo-triangulated category*. We often represent a pseudo-triangulated category simply by \mathcal{C} .

Example 3.4. Let $(\mathcal{C}, \Sigma, \Omega, \triangleright, \triangleleft, \psi)$ be a pseudo-triangulated category.

- (1) \mathcal{C} is an abelian category if and only if $\Sigma = \Omega = 0$.
- (2) \mathcal{C} is a triangulated category if and only if Σ is the quasi-inverse of Ω and ψ is the one induced from the isomorphism $\Sigma \circ \Omega \cong \text{Id}_{\mathcal{C}}$.

Proof. (1) We only show that $\Sigma = \Omega = 0$ implies the abelianess of \mathcal{C} . The converse is confirmed by a routine work. Since $\Sigma = 0$, Proposition 2.4 means $g = \text{cok}(f)$ holds for any object

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

in \triangleright .

Thus (RTR1) implies the existence of a cokernel for each morphism. Dually for the existence of $\ker(f)$. Moreover, in this case f is Σ -null if and only if $f = 0$, and f is Σ -epic if and only if it is epimorphic. Thus (G1) means that any epimorphism g agrees with $\text{cok}(\ker(g))$. Dually for monomorphisms.

(2) In this case, any morphism is at the same time Σ -null and Σ -epic, and Ω -null and Ω -monic. Moreover, \triangleright and \triangleleft agree. We only show $\triangleleft \subseteq \triangleright$.

By (LTR2), for any object

$$(3.1) \quad \Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C$$

in \triangleleft , the shifted one

$$\Omega B \xrightarrow{-\Omega g} \Omega C \xrightarrow{e} A \xrightarrow{f} B$$

is also in \triangleleft . By (G1), we obtain an object in \triangleright

$$\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{\psi(\Omega g)} \Sigma \Omega C,$$

which is isomorphic to (3.1). □

4. EXTENSIONS

In this section, \mathcal{C} is a pseudo-triangulated category with pseudo-triangulation $(\Sigma, \Omega, \triangleright, \triangleleft, \psi)$. We define the notion of an extension which generalizes a short exact sequence in an abelian category, and a distinguished triangle in a triangulated category.

Definition 4.1. A sequence in \mathcal{C}

$$\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is called an *extension* if it satisfies

$$\begin{aligned} (A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A) &\in \triangleright, \\ (\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C) &\in \triangleleft, \\ h &= -\psi_{C,A}(e). \end{aligned}$$

Since e and h determines each other, we sometimes omit one of them.

A *morphism of extensions* from

$$\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

to

$$\Omega C' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

is a triplet (a, b, c) of $a \in \mathcal{C}(A, A')$, $b \in \mathcal{C}(B, B')$ and $c \in \mathcal{C}(C, C')$ satisfying

$$b \circ f = f' \circ a, \quad c \circ g = g' \circ b, \quad (\Sigma a) \circ h = h' \circ c.$$

Remark that $(\Sigma a) \circ h = h' \circ c$ is equivalent to $a \circ e = e' \circ (\Omega c)$. Thus, a morphism of extensions is essentially the same as a morphism in \triangleright or \triangleleft .

$$\begin{array}{ccccccccc} \Omega C & \xrightarrow{e} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \Omega c \downarrow & \circlearrowleft & \downarrow a & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c & \circlearrowleft & \downarrow \Sigma a \\ \Omega C' & \xrightarrow{e'} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

Remark 4.2. Consider a diagram in \mathcal{C}

$$(4.1) \quad \Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

satisfying $h = -\psi(e)$. By (G1) and (G2) (and (RTR1) and (LTR1)), the following are equivalent.

- (1) $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C$ belongs to \triangleleft and g is Σ -epic.
- (2) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ belongs to \triangleright and f is Ω -monic.
- (3) (4.1) is an extension.

Corollary 4.3.

- (1) $g \in \mathcal{C}(B, C)$ is Σ -epic if and only if there exists an extension (4.1), if and only if there exists an object $A \rightarrow B \xrightarrow{g} C \rightarrow \Sigma A$ in \triangleright .
- (2) $f \in \mathcal{C}(A, B)$ is Ω -monic if and only if there exists an extension (4.1), if and only if there exists an object $\Omega C \rightarrow A \xrightarrow{f} B \rightarrow C$ in \triangleleft .

Proof. We show only (1). If there exists an object $A \rightarrow B \xrightarrow{g} C \rightarrow \Sigma A$ in \triangleright , then by (RTR2), we have an object in \triangleright

$$B \xrightarrow{g} C \rightarrow \Sigma A \rightarrow \Sigma B.$$

Obviously this implies g is Σ -epic.

Conversely if g is Σ -epic, then by (LTR1) and Remark 4.2, we obtain an extension (4.1). \square

Lemma 4.4. *Let $f \in \mathcal{C}(A, B)$, $m \in \mathcal{C}(A, M)$ and $e \in \mathcal{C}(M, B)$ be morphisms satisfying $e \circ m = f$.*

- (1) *If f is Σ -epic, then so is e .*
- (2) *If f is Ω -monic, then so is m .*

Proof. (1) By (RTR1) and (RTR3), there exists a morphism in \triangleright

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ m \downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \Sigma m \\ M & \xrightarrow{e} & B & \xrightarrow{g'} & D & \xrightarrow{h'} & \Sigma M. \end{array}$$

Then since g is Σ -null, so is g' . (2) is shown dually. \square

Example 4.5. The notion of an extension becomes as follows in the two cases of Example 3.4.

- (1) If $\Sigma = \Omega = 0$ and \mathcal{C} is abelian, then an extension is nothing other than a short exact sequence.
- (2) If \mathcal{C} is a triangulated category as in Example 3.4, then an extension is nothing other than a distinguished triangle.

Proposition 4.6. *For any $A, B \in \mathcal{C}$,*

$$\Omega B \xrightarrow{0} A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

is an extension, where i_A and p_B are the injection and the projection, respectively.

Proof. Let $p_A: A \oplus B \rightarrow A$ be the projection, and $i_B: B \rightarrow A \oplus B$ be the inclusion. Since id_B is Σ -epic by (RTR1), so is p_B by Lemma 4.4. Thus by Corollary 4.3, there is an extension

$$\Omega B \xrightarrow{u} \exists C \xrightarrow{v} A \oplus B \xrightarrow{p_B} B \xrightarrow{w} \Sigma C$$

with some morphisms u, v, w . Since p_B is the projection and $w \circ p_B = 0$ by Proposition 2.4, we have $w = 0$, and thus $u = 0$. By $p_B \circ i_A = 0$, there exists $r \in \mathcal{C}(A, C)$ such that $v \circ r = i_A$.

$$\begin{array}{ccccc} & & A & & \\ & \swarrow \exists r & \downarrow i_A & \searrow & \\ \Omega B & \xrightarrow{u=0} & C & \xrightarrow{v} & A \oplus B \xrightarrow{p_B} B \end{array}$$

Then we have

$$\begin{aligned} v \circ (\text{id}_C - r \circ (p_A \circ v)) &= v - v \circ r \circ p_A \circ v \\ &= (\text{id}_C - i_A \circ p_A) \circ v \\ &= (i_B \circ p_B) \circ v = 0. \end{aligned}$$

Thus $\text{id}_C - r \circ p_A \circ v$ factors through $u = 0$, which means

$$r \circ (p_A \circ v) = \text{id}_C.$$

Since $(p_A \circ v) \circ r = p_A \circ i_A = \text{id}_A$, this means r is an isomorphism. \square

Proposition 4.7. *Let*

$$\begin{aligned} \Omega C &\xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A, \\ \Omega B' &\xrightarrow{k} A \xrightarrow{\ell} M \xrightarrow{m} B' \xrightarrow{n} \Sigma A, \\ \Omega B &\xrightarrow{k'} A' \xrightarrow{\ell'} M \xrightarrow{m'} B' \xrightarrow{n'} \Sigma A', \end{aligned}$$

be extensions, satisfying $m' \circ \ell = f$. Then there exist $g' \in \mathcal{C}(B', C)$ and $h' \in \mathcal{C}(C, \Sigma A')$ such that

$$\begin{aligned} h' \circ g &= n' \quad , \quad h \circ g' = n, \\ g' \circ m &= g \circ m' \quad , \quad (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0, \end{aligned}$$

and

$$\Omega C \rightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C \xrightarrow{h'} \Sigma A'$$

is an extension. Here we put $f' = m \circ \ell'$. Remark if we put $e' = -\psi^{-1}(h')$, then $(\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0$ is equivalent to $\ell' \circ e' + \ell \circ e = 0$.

Dual statement also holds.

Proof. By (RTR4), there exist $g' \in \mathcal{C}(B', C)$ and $h' \in \mathcal{C}(C, \Sigma A')$ such that

$$\begin{aligned} h' \circ g &= n' \quad , \quad h \circ g' = n, \\ g' \circ m &= g \circ m' \quad , \quad (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0, \end{aligned}$$

and

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C \xrightarrow{h'} \Sigma A'$$

is an object in \triangleright . Thus by Remark 4.2, it suffices to show f' is Ω -monic. This follows from (LTR4). In fact, applying (LTR4) to objects in \triangleleft

$$\begin{aligned} \Omega B &\xrightarrow{-\Omega g} \Omega C \xrightarrow{e} A \xrightarrow{f} B, \\ \Omega B' &\xrightarrow{k} A \xrightarrow{\ell} M \xrightarrow{m} B', \\ \Omega B &\xrightarrow{k'} A' \xrightarrow{\ell'} M \xrightarrow{m'} B, \end{aligned}$$

we obtain an object in \triangleleft

$$\Omega B' \rightarrow \Omega C \rightarrow A' \xrightarrow{f'} B',$$

which means f' is Ω -monic. \square

5. FROBENIUS CONDITION

In this section, we define an extension-closed subcategory \mathcal{Z} of \mathcal{C} , and the Frobenius condition on it. This condition generalizes simultaneously the usual Frobenius condition for an exact category, and the existence of a subcategory \mathcal{D} such that $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair in the case of a triangulated category.

Definition 5.1. A subcategory $\mathcal{Z} \subseteq \mathcal{C}$ is said to be *extension-closed* if it satisfies the following.

(*) For any extension in \mathcal{C}

$$\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

$X, Z \in \mathcal{Z}$ implies $Y \in \mathcal{Z}$.

In the following, we fix an extension-closed subcategory $\mathcal{Z} \subseteq \mathcal{C}$.

Remark 5.2. When \mathcal{C} is an abelian category as in Example 4.5, then \mathcal{Z} is an exact category.

Definition 5.3. Let $\mathcal{Z} \subseteq \mathcal{C}$ be an extension-closed subcategory as above.

(1) A *conflation* is an extension in \mathcal{C}

$$(5.1) \quad \Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

satisfying $X, Y, Z \in \mathcal{Z}$. A *morphism of conflations* is a morphism of the extensions.

(2) A morphism $f: X \rightarrow Y$ in \mathcal{Z} is an *inflation* if there exists a conflation (5.1).

(3) A morphism $g: Y \rightarrow Z$ in \mathcal{Z} is a *deflation* if there exists a conflation (5.1).

In the following, we fix an extension-closed subcategory $\mathcal{Z} \subseteq \mathcal{C}$. For a full additive replete subcategory $\mathcal{D} \subseteq \mathcal{Z}$, we consider the following condition (DS).

Condition 5.4.

(DS) \mathcal{D} is *closed under finite direct summands in \mathcal{Z}* , namely, for any $Z_1, Z_2 \in \mathcal{Z}$ and $D \in \mathcal{D}$, $D \cong Z_1 \oplus Z_2$ implies $Z_1, Z_2 \in \mathcal{Z}$.

Definition 5.5. Let $\mathcal{D} \subseteq \mathcal{Z}$ be a full additive replete subcategory satisfying (DS).

(1) An object I in \mathcal{D} is *injective* if

$$\mathcal{Z}(Y, I) \xrightarrow{- \circ f} \mathcal{Z}(X, I) \rightarrow 0$$

is exact for any inflation $f: X \rightarrow Y$. We denote the full subcategory of injective objects by $\mathcal{I}_{\mathcal{D}} \subseteq \mathcal{D}$. In particular $\mathcal{I}_{\mathcal{Z}}$ is denoted by \mathcal{I} .

(2) An object P in \mathcal{D} is *projective* if

$$\mathcal{Z}(P, Y) \xrightarrow{g \circ -} \mathcal{Z}(P, Z) \rightarrow 0$$

is exact for any deflation $g: Y \rightarrow Z$. We denote the full subcategory of projective objects by $\mathcal{P}_{\mathcal{D}} \subseteq \mathcal{D}$. In particular $\mathcal{P}_{\mathcal{Z}}$ is denoted by \mathcal{P} .

Example 5.6.

- (1) If $\mathcal{Z} \subseteq \mathcal{C}$ is an exact category where \mathcal{C} is an abelian category as in Example 4.5, then \mathcal{I} is equal to the full subcategory of injective objects, and \mathcal{P} is equal to the full subcategory of projective objects.
- (2) If \mathcal{C} is a triangulated category, and if \mathcal{D} satisfies $\mathcal{C}(\Omega\mathcal{Z}, \mathcal{D}) = \mathcal{C}(\mathcal{D}, \Sigma\mathcal{Z}) = 0$, then we have $\mathcal{I}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}} = \mathcal{D}$.

Caution 5.7. The definitions of injective and projective objects are different from those in [B].

Remark 5.8.

- (1) $\mathcal{I}_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{D}}$ are full additive replete subcategories, which are closed under finite direct summands in \mathcal{Z} .
- (2) $\mathcal{I}_{\mathcal{D}} = \mathcal{I} \cap \mathcal{D}$.
- (3) $\mathcal{P}_{\mathcal{D}} = \mathcal{P} \cap \mathcal{D}$.

Proof. Left to the reader. □

Definition 5.9. Let $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ be a triplet as above.

- (1) $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ *has enough injectives* if for any $X \in \mathcal{Z}$, there exists an inflation $\alpha: X \rightarrow I$ such that $I \in \mathcal{I}_{\mathcal{D}}$. When $\mathcal{D} = \mathcal{Z}$, we simply say “ \mathcal{Z} has enough injectives”.
- (2) $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ *has enough projectives* if for any $Z \in \mathcal{Z}$, there exists a deflation $\beta: P \rightarrow Z$ such that $P \in \mathcal{P}_{\mathcal{D}}$. When $\mathcal{D} = \mathcal{Z}$, we simply say “ \mathcal{Z} has enough projectives”.
- (3) $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is *Frobenius* if it has enough injectives and projectives, and moreover $\mathcal{I}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}}$. When $\mathcal{D} = \mathcal{Z}$, we simply say “ \mathcal{Z} is Frobenius”.

Example 5.10.

- (1) If $\mathcal{Z} \subseteq \mathcal{C}$ is an exact category as in Example 5.6, then \mathcal{Z} is Frobenius if and only if \mathcal{Z} is Frobenius as an exact category. In this case the stable category \mathcal{Z}/\mathcal{I} is triangulated [H].
- (2) If \mathcal{C} is a triangulated category and if $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair in \mathcal{C} (in the definition in [IY]), then $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius. In this case $\mathcal{Z}/\mathcal{I}_{\mathcal{D}} = \mathcal{Z}/\mathcal{D}$ becomes a triangulated category by Theorem 4.2 in [IY].

	Happel’s construction [H]	Iyama and Yoshino’s construction [IY]
\mathcal{C}	abelian category	triangulated category
\mathcal{Z}	exact subcategory	extension-closed subcategory
\mathcal{D}	$\mathcal{Z} = \mathcal{D}$	$(\mathcal{Z}, \mathcal{Z}) : \mathcal{D}$ -mutation pair
$\mathcal{I}_{\mathcal{D}}$	injective objects	$\mathcal{I}_{\mathcal{D}} = \mathcal{D}$
$\mathcal{P}_{\mathcal{D}}$	projective objects	$\mathcal{P}_{\mathcal{D}} = \mathcal{D}$

In section 6, in a pseudo-triangulated category \mathcal{C} satisfying Condition 6.1, we show $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$ becomes a triangulated category for any Frobenius triplet $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ (Theorem 6.17), which we call the *stable category* associated to $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$. In particular, if \mathcal{Z} is Frobenius, then \mathcal{Z}/\mathcal{I} becomes a triangulated category. We call \mathcal{Z}/\mathcal{I} the stable category associated to \mathcal{Z} .

Although we have defined the Frobenius condition on a triplet $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$, it is essentially the same as the Frobenius condition on \mathcal{Z} as follows (Corollary 5.13).

Proposition 5.11. *Let $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{Z}$ be full additive replete subcategories satisfying (DS). If $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius, so is $(\mathcal{C}, \mathcal{Z}, \mathcal{D}')$. Moreover, we have $\mathcal{I}_{\mathcal{D}'} = \mathcal{I}_{\mathcal{D}}$.*

Proof. This immediately follows from the lemma below. \square

Lemma 5.12. *Let $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{Z}$ be as in Proposition 5.11. If $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ has enough injectives, then we have $\mathcal{I}_{\mathcal{D}'} = \mathcal{I}_{\mathcal{D}}$. Similarly for projectives.*

Proof. Remark that $\mathcal{I}_{\mathcal{D}} = \mathcal{I}_{\mathcal{D}'} \cap \mathcal{D}$. Thus it suffices to show $\mathcal{I}_{\mathcal{D}'} \subseteq \mathcal{D}$.

Since $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ has enough injectives, for any $I' \in \mathcal{I}_{\mathcal{D}'}$, there exists a conflation

$$\Omega Z \xrightarrow{e} I' \xrightarrow{f} I \xrightarrow{g} Z \xrightarrow{h} \Sigma I',$$

where $Z \in \mathcal{Z}$ and $I \in \mathcal{I}_{\mathcal{D}}$. Since $I' \in \mathcal{I}_{\mathcal{D}'}$, there exists $p \in \mathcal{Z}(I, I')$ such that $p \circ f = \text{id}_{I'}$. By $f \circ e = 0$, we have $e = p \circ f \circ e = 0$, and thus $h = 0$. By $(\text{id}_I - f \circ p) \circ f = 0$, there exists $s \in \mathcal{Z}(Z, I)$ such that $s \circ g = \text{id}_I - f \circ p$. Since $(\text{id}_Z - g \circ s) \circ g = 0$, $\text{id}_Z - g \circ s$ factors through $h = 0$, namely, we have $\text{id}_Z = g \circ s$. Thus we obtain $I = I' \oplus Z$. Since \mathcal{D} is closed under finite direct summands in \mathcal{Z} , it follows $I' \in \mathcal{D}$. \square

Thus if $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is a Frobenius triplet, then \mathcal{Z} is Frobenius, and satisfies $\mathcal{I} = \mathcal{I}_{\mathcal{D}}$. In particular, their stable categories are equivalent.

Corollary 5.13. *For any extension-closed subcategory $\mathcal{Z} \subseteq \mathcal{C}$, the following are equivalent.*

- (1) \mathcal{Z} is Frobenius.
- (2) There exists a full additive replete subcategory $\mathcal{D} \subseteq \mathcal{Z}$ satisfying (DS) such that $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius.

Moreover, there exists the minimum one.

Corollary 5.14. *If \mathcal{Z} is Frobenius, there exists the minimum \mathcal{D} , which makes $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ Frobenius.*

Proof. We show \mathcal{I} satisfies the desired conditions. By Remark 5.8, $\mathcal{I} \subseteq \mathcal{Z}$ is a full additive replete subcategory satisfying (DS). If \mathcal{Z} is Frobenius, it immediately follows that

$$\mathcal{I}_{\mathcal{I}} = \mathcal{I} = \mathcal{P} = \mathcal{P}_{\mathcal{I}},$$

and $(\mathcal{C}, \mathcal{Z}, \mathcal{I})$ becomes Frobenius. Obviously \mathcal{I} is the minimum one, since any Frobenius triplet $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ satisfies $\mathcal{I} = \mathcal{I}_{\mathcal{D}} \subseteq \mathcal{D}$. \square

When \mathcal{C} is a triangulated category and if $\mathcal{D} \subseteq \mathcal{Z}$ is a full additive replete subcategory satisfying (DS) and

$$\mathcal{C}(\Omega \mathcal{Z}, \mathcal{D}) = \mathcal{C}(\mathcal{D}, \Sigma \mathcal{Z}) = 0,$$

then $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius if and only if $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair. (We also remark that if there exists one such \mathcal{D} , then it is unique and must agree with the full subcategory of \mathcal{Z} consisting of those $D \in \mathcal{Z}$ satisfying $\mathcal{C}(\Omega \mathcal{Z}, D) = \mathcal{C}(D, \Sigma \mathcal{Z}) = 0$.)

Namely, we have the following.

Claim 5.15. *Let $\mathcal{D} \subseteq \mathcal{Z}$ be a full additive replete subcategory satisfying (DS). The following are equivalent.*

- (1) $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius, and $\mathcal{C}(\Omega \mathcal{Z}, \mathcal{D}) = \mathcal{C}(\mathcal{D}, \Sigma \mathcal{Z}) = 0$.
- (2) $(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{D} -mutation pair.

Regarding Corollary 5.13 and Corollary 5.14, we obtain the following.

Corollary 5.16. *For any \mathcal{Z} , the following are equivalent.*

- (1) \mathcal{Z} is Frobenius, and $\mathcal{C}(\Omega\mathcal{Z}, \mathcal{I}) = \mathcal{C}(\mathcal{I}, \Sigma\mathcal{Z}) = 0$.
- (2) $(\mathcal{Z}, \mathcal{Z})$ is an \mathcal{I} -mutation pair.

6. TRIANGULATION ON THE STABLE CATEGORY

In this section, as a main theorem, we show give a triangulation on the stable category associated to an extension-closed subcategory of a pseudo-triangulated category satisfying the following condition. Remark that this condition is trivially satisfied in the two cases of Example 3.4.

Condition 6.1. Let

$$\begin{array}{ccccccc} \Omega C & \xrightarrow{e} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \xrightarrow{h} \Sigma A, \\ \Omega C' & \xrightarrow{e'} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \xrightarrow{h'} \Sigma A' \end{array}$$

be extensions.

- (AC1) If $c \in \mathcal{C}(C, C')$ satisfies $h' \circ c = 0$ and $c \circ g = 0$, then there exists $c' \in \mathcal{C}(C, B')$ such that $g' \circ c' = c$.
- (AC2) If $a \in \mathcal{C}(A, A')$ satisfies $f' \circ a = 0$ and $a \circ e = 0$, then there exists $a' \in \mathcal{C}(B, A')$ such that $a' \circ f = a$.

Remark 6.2. If we impose the following conditions (1) and (2) on \mathcal{C} (cf. [BR]), then Condition 6.1 is satisfied.

- (1) There exists an adjoint natural isomorphism

$$\varphi_{A,B}: \mathcal{C}(\Sigma A, B) \xrightarrow{\cong} \mathcal{C}(A, \Omega B) \quad (A, B \in \mathcal{C}).$$

- (2) Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ and $\Omega C' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ be any object in \triangleright and \triangleleft , respectively.

For any $a \in \mathcal{C}(A, \Omega C')$ and $b \in \mathcal{C}(B, A')$ satisfying $b \circ f = e' \circ a$, there exists $c \in \mathcal{C}(C, B')$ such that $c \circ g = f' \circ b$ and $\varphi_{A,C'}^{-1}(a) \circ h = g' \circ c$.

For any $c \in \mathcal{C}(C, B')$ and $d \in \mathcal{C}(\Sigma A, C')$ satisfying $d \circ h = g' \circ c$, there exists $b \in \mathcal{C}(B, A')$ such that $c \circ g = f' \circ b$ and $b \circ f = e' \circ \varphi_{A,C'}(d)$.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow & \circ & \downarrow & \circ & \downarrow & \circ & \downarrow \\ \Omega C' & \xrightarrow{e'} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

In the rest, \mathcal{C} is assumed to satisfy Condition 5.14. First, we construct the shift functor.

Lemma 6.3. *Let*

$$\begin{array}{ccccccc} \Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \downarrow & \circ & \downarrow & \circ & \downarrow & \circ & \downarrow \\ \Omega S & \xrightarrow{\delta} & M & \xrightarrow{\alpha} & I & \xrightarrow{\beta} & S \xrightarrow{\gamma} \Sigma M \end{array}, \quad \begin{array}{ccccccc} \Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \downarrow & \circ & \downarrow & \circ & \downarrow & \circ & \downarrow \\ \Omega S & \xrightarrow{\delta} & M & \xrightarrow{\alpha} & I & \xrightarrow{\beta} & S \xrightarrow{\gamma} \Sigma M \end{array}$$

be morphisms of conflations, with $I \in \mathcal{I}_{\mathcal{D}}$. Then $\underline{x} = \underline{x}'$ in $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$ implies $\underline{z} = \underline{z}'$ in $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$.

Proof. Obviously, it suffices to show that $\underline{x} = 0$ implies $\underline{z} = 0$ in the first diagram.

Since $\underline{x} = 0$, there exist $I_0 \in \mathcal{I}_{\mathcal{D}}$, $x_1 \in \mathcal{Z}(X, I_0)$ and $x_2 \in \mathcal{Z}(I_0, M)$ such that $x = x_2 \circ x_1$. Since $I_0 \in \mathcal{I}_{\mathcal{D}}$ and f is an inflation, there exists $x_3 \in \mathcal{Z}(Y, I_0)$ such that $x_3 \circ f = x_1$. Thus we have $x \circ e = x_2 \circ x_3 \circ f \circ e = 0$, which implies

$$(\Sigma x) \circ h = -(\Sigma x) \circ \psi(e) = -\psi(x \circ e) = 0.$$

Put $\eta = y - \alpha \circ x_2 \circ x_3$. By $\eta \circ f = 0$, there exists $s \in \mathcal{Z}(Z, I)$ such that $s \circ g = \eta$. Thus we have

$$\begin{aligned} \gamma \circ (z - \beta \circ s) &= \gamma \circ z = (\Sigma x) \circ h = 0, \\ (z - \beta \circ s) \circ g &= z \circ g - \beta \circ y = 0. \end{aligned}$$

By (AC1), there exists $t \in \mathcal{Z}(Z, I)$ such that $z - \beta \circ s = \beta \circ t$, namely $z = \beta \circ (s + t)$. \square

Construction 6.4. Assume $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ has enough injectives. For any $X \in \mathcal{Z}$, take a conflation

$$\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$$

with $I_X \in \mathcal{I}_{\mathcal{D}}$. Define $S(X) = SX$ to be the image of S_X in $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$.

For any morphism $f \in \mathcal{Z}(X, Y)$, take a conflation

$$\Omega S_Y \xrightarrow{\delta_Y} Y \xrightarrow{\alpha_Y} I_Y \xrightarrow{\beta_Y} S_Y \xrightarrow{\gamma_Y} \Sigma Y$$

similarly for Y . Since α_X is an inflation and $I_Y \in \mathcal{I}_{\mathcal{D}}$, there exists $I_f \in \mathcal{Z}(I_X, I_Y)$ such that $I_f \circ \alpha_X = \alpha_Y \circ f$. By (RTR3), there exists $S_f \in \mathcal{Z}(S_X, S_Y)$ such that (f, I_f, S_f) is a morphism of conflations.

$$\begin{array}{ccccccccc} \Omega S_X & \xrightarrow{\delta_X} & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X & \xrightarrow{\gamma_X} & \Sigma X \\ \Omega S_f \downarrow & \circ & \downarrow f & \circ & \downarrow I_f & \circ & \downarrow S_f & \circ & \downarrow \Sigma f \\ \Omega S_Y & \xrightarrow{\delta_Y} & Y & \xrightarrow{\alpha_Y} & I_Y & \xrightarrow{\beta_Y} & S_Y & \xrightarrow{\gamma_Y} & \Sigma Y \end{array}$$

For any $\underline{f} \in \mathcal{Z}/\mathcal{I}_{\mathcal{D}}(X, Y)$, define \underline{Sf} to be the image $\underline{S_f}$ of S_f in $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$. This is well-defined by Lemma 6.3, and the following proposition holds.

Proposition 6.5. $S: \mathcal{Z}/\mathcal{I}_{\mathcal{D}} \rightarrow \mathcal{Z}/\mathcal{I}_{\mathcal{D}}$ gives an additive functor.

Proof. This immediately follows from Lemma 6.3. \square

Remark 6.6. Dually, if $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ has enough projectives, then we have an additive functor $S^*: \mathcal{Z}/\mathcal{P}_{\mathcal{D}} \rightarrow \mathcal{Z}/\mathcal{P}_{\mathcal{D}}$, defined by a conflation

$$\Omega X \rightarrow S^* X \rightarrow P_X \rightarrow X \rightarrow \Sigma S^* X$$

for any $X \in \mathcal{Z}$, where $P_X \in \mathcal{P}_{\mathcal{D}}$.

Proposition 6.7. If $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius, then S and S^* are quasi-inverses.

Proof. This follows immediately from the definitions of S and S^* . \square

In the rest, $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is assumed to be Frobenius. Next, we define the class of distinguished triangles on $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$.

Definition 6.8. Let $\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be any conflation, and take a conflation $\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$ where $I_X \in \mathcal{I}_D$.

If there exist $p \in \mathcal{Z}(Y, I_X)$ and $q \in \mathcal{Z}(Z, S_X)$ satisfying

$$p \circ f = \alpha_X, \quad q \circ g = \beta_X \circ p, \quad \gamma_X \circ q = h$$

(namely, (id, p, q) is a morphism of conflations)

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \parallel & \circlearrowleft & \downarrow p & \circlearrowleft & \downarrow q & \circlearrowleft & \parallel \\ X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X & \xrightarrow{\gamma_X} & \Sigma X \end{array},$$

then we call the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} S_X$$

a *standard triangle*. Remark that by (RTR3) and the injectivity of I_X , there exists at least one such pair of morphisms (p, q) . We define the class of distinguished triangles Δ to be the category of triangles

$$(6.1) \quad X \rightarrow Y \rightarrow Z \rightarrow SZ$$

in $\mathcal{Z}/\mathcal{I}_D$, which are isomorphic to standard triangles.

In the rest, we show that $(\mathcal{Z}/\mathcal{I}_D, S, \Delta)$ is a triangulated category.

Proposition 6.9. $(\mathcal{Z}/\mathcal{I}_D, S, \Delta)$ satisfies (TR1).

Proof.

- (1) By definition, every diagram (6.1) isomorphic to an object in Δ also belongs to Δ .
- (2) Let $f \in \mathcal{Z}(X, Y)$ be any morphism. Take a conflation

$$\Omega S_X \xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X$$

with $I_X \in \mathcal{I}_D$, and put $f_X = (f, -\alpha_X)$. By Corollary 4.3, Lemma 4.4 and Proposition 4.7, $f_X: X \rightarrow Y \oplus I_X$ becomes an inflation. In fact, by Corollary 4.3 and Lemma 4.4, there exists an extension

$$\Omega C_f \rightarrow X \xrightarrow{f_X} Y \oplus I_X \xrightarrow{c_f} C_f \xrightarrow{\ell_f} \Sigma X,$$

and applying Proposition 4.7 to the following diagram (6.2) of extensions, we obtain an extension

$$\Omega S_X \rightarrow Y \rightarrow C_f \xrightarrow{\exists q} S_X \rightarrow \Sigma Y,$$

and thus $C_f \in \mathcal{Z}$ by the extension-closedness of \mathcal{Z} .

$$(6.2) \quad \begin{array}{ccccccc} & & & \Sigma Y & & & \\ & & & \uparrow & \swarrow & & \\ & & & I_X & \xrightarrow{\beta_X} & S_X & \\ & \nearrow \alpha_X & & \uparrow & \circlearrowleft & \uparrow \exists q & \searrow \gamma_X \\ \Omega C_f & \longrightarrow & X & \xrightarrow{f_X} & Y \oplus I_X & \xrightarrow{c_f} & C_f \xrightarrow{\ell_f} \Sigma X \\ & \uparrow & \uparrow i_Y & \uparrow & \circlearrowleft & \uparrow & \\ & \Omega S_X & \dashrightarrow & Y & \dashrightarrow & C_f & \\ & \uparrow & & \uparrow & & & \\ & \Omega I_X & & & & & \end{array}$$

Let $C(f)$ denote the image of C_f in $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$. Then the above diagram means

$$X \xrightarrow{f_X} Y \oplus I_X \xrightarrow{c_f} C(f) \xrightarrow{q} SX$$

is a standard triangle. If we put $g = c_f \circ i_Y$ where $i_Y: Y \hookrightarrow Y \oplus I_X$ is the inclusion, then $X \xrightarrow{f} Y \xrightarrow{g} C(f) \xrightarrow{q} SX$ becomes isomorphic to this standard triangle.

(3) By (RTR1), (RTR2) and (LTR1),

$$0 = \Omega 0 \rightarrow X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow \Sigma X$$

is a conflation, and it immediately follows that the triangle

$$X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow SX$$

belongs to Δ .

□

Proposition 6.10. $(\mathcal{Z}/\mathcal{I}_{\mathcal{D}}, S, \Delta)$ satisfies (TR2).

Proof. It suffices to show, for any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} SX$$

arising from a morphism of conflations

$$\begin{array}{ccccccc} \Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \Omega q \downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow p & \circlearrowleft & \downarrow q \circlearrowleft \parallel \\ \Omega S_X & \xrightarrow{\delta_X} & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X \xrightarrow{\gamma_X} \Sigma X \end{array},$$

the shifted triangle

$$Y \xrightarrow{g} Z \xrightarrow{q} SX \xrightarrow{-Sf} SY$$

also becomes a distinguished triangle.

We may replace $\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ by the conflation $\Omega C_f \rightarrow X \xrightarrow{f_X} Y \oplus I_X \xrightarrow{c_f} C_f \xrightarrow{\ell_f} \Sigma X$ constructed in the proof of Proposition 6.9. Recall that $f_X = (f, -\alpha_X) = i_Y \circ f - i_{I_X} \circ \alpha_X$ where i_Y and i_{I_X} are the inclusions into $Y \oplus I_X$.

Take conflations

$$\begin{aligned} \Omega S_X &\xrightarrow{\delta_X} X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X, \\ \Omega I_X &\xrightarrow{0} Y \xrightarrow{i_Y} Y \oplus I_X \xrightarrow{-p_{I_X}} I_X \xrightarrow{0} \Sigma Y. \end{aligned}$$

By Proposition 4.7, there exists $k \in \mathcal{C}(\Omega S_X, Y)$ and $\nu \in \mathcal{Z}(C_f, S_X)$ such that

$$\Omega S_X \xrightarrow{k} Y \xrightarrow{\mu} C_f \xrightarrow{\nu} S_X \rightarrow \Sigma Y$$

is a conflation, where $\mu = c_f \circ i_Y$, and

$$\begin{aligned} \nu \circ c_f &= -\beta_X \circ p_{I_X} & , & \quad \gamma_X \circ \nu = \ell_f, \\ -\psi_{S_X, Y}(k) \circ \beta_X &= 0 & , & \quad f_X \circ \delta_X + i_Y \circ k = 0. \end{aligned}$$

$$\begin{array}{ccccccc} & & & & \Sigma Y & & \\ & & & & \uparrow 0 & \swarrow -\psi_{S_X, Y}(k) & \\ & & & & I_X & \xrightarrow{\beta_X} & S_X \\ & & \nearrow \alpha_X & \uparrow -p_{I_X} & \uparrow \exists \nu & \nearrow \gamma_X & \\ \Omega C_f & \longrightarrow & X & \xrightarrow{f_X} & Y \oplus I_X & \xrightarrow{c_f} & C_f \xrightarrow{\ell_f} \Sigma X \\ & & \uparrow \delta_X & \uparrow i_Y & \uparrow \mu & & \\ & & \Omega S_X & \xrightarrow{-k} & Y & & \\ & & \uparrow 0 & & \Omega I_X & & \end{array}$$

Claim 6.11. *We have a morphism of conflations*

$$\begin{array}{ccccccc} \Omega S_X & \xrightarrow{\delta_X} & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X \xrightarrow{\gamma_X} \Sigma X \\ -\text{id} \downarrow & \circlearrowleft & \downarrow f & \circlearrowleft & \downarrow c_f \circ i_{I_X} & \circlearrowleft & \downarrow -\text{id} \circlearrowleft \downarrow \Sigma f \\ \Omega S_X & \xrightarrow{k} & Y & \xrightarrow{\mu} & C_f & \xrightarrow{\nu} & S_X \longrightarrow \Sigma Y \end{array}$$

Proof of Claim 6.11. This immediately follows from

$$f \circ \delta_X = p_Y \circ f_X \circ \delta_X = -p_Y \circ i_Y \circ k = -k,$$

$$\begin{aligned} c_f \circ i_{I_X} \circ \alpha_X &= c_f \circ i_{I_X} \circ (-p_{I_X}) \circ f_X \\ &= c_f \circ (i_Y \circ p_Y \circ f_X - f_X) \\ &= c_f \circ i_Y \circ f, \end{aligned}$$

$$\nu \circ c_f \circ i_{I_X} = -\beta_X \circ p_{I_X} \circ i_{I_X} = -\beta_X.$$

□

If we take a conflation $\Omega S_Y \xrightarrow{\delta_Y} Y \xrightarrow{\alpha_Y} I_Y \xrightarrow{\beta_Y} S_Y \xrightarrow{\gamma_Y} \Sigma Y$ where $I_Y \in \mathcal{I}_{\mathcal{D}}$, then there exist $u \in \mathcal{Z}(C_f, I_Y)$ and $v \in \mathcal{Z}(S_X, S_Y)$ such that (id_Y, p, q) is a morphism

of conflations.

$$(6.3) \quad \begin{array}{ccccccccc} \Omega S_X & \xrightarrow{k} & Y & \xrightarrow{\mu} & C_f & \xrightarrow{\nu} & S_X & \longrightarrow & \Sigma Y \\ \Omega q \downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow u & \circlearrowleft & \downarrow v & \circlearrowleft & \parallel \\ \Omega S_Y & \xrightarrow{\delta_Y} & Y & \xrightarrow{\alpha_Y} & I_Y & \xrightarrow{\beta_Y} & S_Y & \xrightarrow{\gamma_Y} & \Sigma Y \end{array}$$

By definition, we have a standard triangle in \triangle

$$Y \xrightarrow{\mu} C(f) \xrightarrow{\nu} S_X \xrightarrow{v} SY.$$

Composing (6.3) with the morphism obtained in Claim 6.11, we obtain the following morphism of conflations, which means $S\underline{f} = -\underline{v}$.

$$\begin{array}{ccccccccc} \Omega S_X & \xrightarrow{\delta_X} & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X & \xrightarrow{\gamma_X} & \Sigma X \\ -\Omega q \downarrow & \circlearrowleft & \downarrow f & \circlearrowleft & \downarrow u \circ c_f \circ i_{I_X} & \circlearrowleft & \downarrow -v & \circlearrowleft & \downarrow \Sigma f \\ \Omega S_Y & \xrightarrow{\delta_Y} & Y & \xrightarrow{\alpha_Y} & I_Y & \xrightarrow{\beta_Y} & S_Y & \xrightarrow{\gamma_Y} & \Sigma Y \end{array}$$

□

Lemma 6.12. *Let*

$$X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{q}} SX$$

and

$$X' \xrightarrow{\underline{f}'} Y' \xrightarrow{\underline{g}'} Z' \xrightarrow{\underline{q}'} SX'$$

be standard triangles in $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$ obtained from

$$\begin{array}{ccccccc} \Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow p & \circlearrowleft & \downarrow q \circlearrowleft \parallel \\ \Omega S_X & \rightarrow & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X \xrightarrow{\gamma_X} \Sigma X \end{array} \quad \text{and} \quad \begin{array}{ccccccc} \Omega Z' & \xrightarrow{e'} & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \xrightarrow{h'} \Sigma X' \\ \downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow p' & \circlearrowleft & \downarrow q' \circlearrowleft \parallel \\ \Omega S_{X'} & \rightarrow & X' & \xrightarrow{\alpha_{X'}} & I_{X'} & \xrightarrow{\beta_{X'}} & S_{X'} \xrightarrow{\gamma_{X'}} \Sigma X' \end{array}.$$

If we are given a morphism of conflations

$$\begin{array}{ccccccc} \Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \Omega z \downarrow & \circlearrowleft & \downarrow x & \circlearrowleft & \downarrow y & \circlearrowleft & \downarrow z \circlearrowleft \downarrow \Sigma x, \\ \Omega Z' & \xrightarrow{e'} & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \xrightarrow{h'} \Sigma X' \end{array}$$

then we obtain the following morphism in \triangle .

$$\begin{array}{ccccccc} X & \xrightarrow{\underline{f}} & Y & \xrightarrow{\underline{g}} & Z & \xrightarrow{\underline{q}} & SX \\ \underline{x} \downarrow & \circlearrowleft & \downarrow \underline{y} & \circlearrowleft & \downarrow \underline{z} & \circlearrowleft & \downarrow S\underline{x} \\ X' & \xrightarrow{\underline{f}'} & Y' & \xrightarrow{\underline{g}'} & Z' & \xrightarrow{\underline{q}'} & SX' \end{array}$$

Proof. It suffices to show $\underline{q}' \circ \underline{z} = (S\underline{x}) \circ \underline{q}$. By the definition of S_x , we have $(\Sigma x) \circ \gamma_X = \gamma_{X'} \circ S_x$. Since $(\underline{I}_x \circ p - p' \circ y) \circ f = \underline{I}_x \circ p \circ f - p' \circ y \circ f = \underline{I}_x \circ \alpha_X - \alpha_{X'} \circ x = 0$

$$\begin{array}{ccccccc} \Omega S_X & \longrightarrow & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X \xrightarrow{\gamma_X} \Sigma X \\ \downarrow & \circlearrowleft & \downarrow x & \circlearrowleft & \downarrow I_x & \circlearrowleft & \downarrow S_x \circlearrowleft \downarrow \Sigma x, \\ \Omega S_{X'} & \longrightarrow & X' & \xrightarrow{\alpha_{X'}} & I_{X'} & \xrightarrow{\beta_{X'}} & S_{X'} \xrightarrow{\gamma_{X'}} \Sigma M \end{array}$$

there exists $s \in \mathcal{Z}(Z, I_{X'})$ such that $s \circ g = \underline{I}_x \circ p - p' \circ y$. If we put $\zeta = S_x \circ q - q' \circ z - \beta_{X'} \circ s$, then ζ satisfies

$$\begin{aligned} \gamma_{X'} \circ \zeta &= \gamma_{X'} \circ S_x \circ q - \gamma_{X'} \circ q' \circ z - \gamma_{X'} \circ \beta_{X'} \circ s \\ &= (\Sigma x) \circ \gamma_X \circ q - h' \circ z \\ &= (\Sigma x) \circ h - (\Sigma x) \circ h \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \zeta \circ g &= S_x \circ q \circ g - q' \circ z \circ g - \beta_{X'} \circ s \circ g \\ &= S_x \circ \beta_X \circ p - q' \circ g' \circ y - (\beta_{X'} \circ \underline{I}_x \circ p - \beta_{X'} \circ p' \circ y) \\ &= \beta_{X'} \circ \underline{I}_x \circ p - \beta_{X'} \circ p' \circ y - (\beta_{X'} \circ \underline{I}_x \circ p - \beta_{X'} \circ p' \circ y) \\ &= 0. \end{aligned}$$

Thus by (AC1), there exists $t \in \mathcal{Z}(Z, I_{X'})$ such that $\zeta = \beta_{X'} \circ t$, i.e.,

$$S_x \circ q - q' \circ z = \beta_{X'} \circ (s + t).$$

$$\begin{array}{ccccccc} \Omega Z & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ & & & & \exists_t \swarrow \circlearrowleft \downarrow \zeta & \circlearrowleft & \downarrow 0 \\ \Omega S_{X'} & \longrightarrow & X' & \xrightarrow{\alpha_{X'}} & I_{X'} & \xrightarrow{\beta_{X'}} & S_{X'} \xrightarrow{\gamma_{X'}} \Sigma X \end{array}$$

□

Proposition 6.13. $(\mathcal{Z}/\mathcal{I}_D, S, \triangle)$ satisfies (TR3).

Proof. Suppose we are given distinguished triangles

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{q} SX \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \xrightarrow{q'} SX' \end{array}$$

and morphisms $x \in \mathcal{Z}(X, X')$ and $y \in \mathcal{Z}(Y, Y')$ satisfying $\underline{y} \circ \underline{f} = \underline{f'} \circ \underline{x}$. We want to find $z \in \mathcal{Z}(Z, Z')$ which satisfies $\underline{z} \circ \underline{g} = \underline{g'} \circ \underline{y}$ and $S\underline{x} \circ \underline{q} = \underline{q'} \circ \underline{z}$.

We may assume these triangles are standard, arising from morphisms of conflations:

$$\begin{array}{ccccccc} \Omega Z & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow p & \circlearrowleft & \downarrow q \circlearrowleft \parallel \\ \Omega S_X & \longrightarrow & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X \xrightarrow{\gamma_X} \Sigma X \end{array}$$

$$\begin{array}{ccccccccc}
\Omega Z' & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \\
\downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow p' & \circlearrowleft & \downarrow q' & \circlearrowleft & \parallel \\
\Omega S_{X'} & \longrightarrow & X' & \xrightarrow{\alpha_{X'}} & I_{X'} & \xrightarrow{\beta_{X'}} & S_{X'} & \xrightarrow{\gamma_{X'}} & \Sigma X'
\end{array}$$

Since $\underline{y} \circ \underline{f} = \underline{f}' \circ \underline{x}$, there exist $I \in \mathcal{I}_{\mathcal{D}}$, $s_1 \in \mathcal{Z}(X, I)$ and $s_2 \in \mathcal{Z}(I, Y')$ such that $s_2 \circ s_1 = \underline{y} \circ \underline{f} - \underline{f}' \circ \underline{x}$. By the injectivity of I , there exists $s_3 \in \mathcal{Z}(Y, I)$ such that $s_3 \circ \underline{f} = s_1$. Then we have $(\underline{y} - s_2 \circ s_3) \circ \underline{f} = \underline{f}' \circ \underline{x}$, and there exists $z \in \mathcal{Z}(Z, Z')$ such that $z \circ \underline{g} = \underline{g}' \circ (\underline{y} - s_2 \circ s_3)$ and $(\Sigma x) \circ h = h' \circ z$ by (RTR3). Thus Proposition 6.13 follows from Lemma 6.12.

$$\begin{array}{ccccccccc}
\Omega Z & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\Omega z \downarrow & \circlearrowleft & x \downarrow & \circlearrowleft & y - s_2 \circ s_3 \downarrow & \circlearrowleft & z \downarrow & \circlearrowleft & \downarrow \Sigma x \\
\Omega S_X & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'
\end{array}$$

□

Proposition 6.14. $(\mathcal{Z}/\mathcal{I}_{\mathcal{D}}, S, \triangle)$ satisfies (TR4).

Proof. Let

$$(6.4) \quad X \xrightarrow{\underline{\ell}} M \xrightarrow{\underline{m}} Y' \xrightarrow{\underline{v}} SX$$

$$(6.5) \quad X' \xrightarrow{\underline{\ell}'} M \xrightarrow{\underline{m}'} Y \xrightarrow{\underline{v}'} SX'$$

$$(6.6) \quad X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{q}} SX,$$

be distinguished triangles in $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$ satisfying $\underline{m}' \circ \underline{\ell} = \underline{f}$. It suffices to show there exist $\underline{g}' \in \mathcal{Z}(Y', Z)$ and $\underline{q}' \in \mathcal{Z}(Z, S_{X'})$ such that

$$X' \xrightarrow{\underline{f}'} Y' \xrightarrow{\underline{g}'} Z \xrightarrow{\underline{q}'} SX'$$

is a standard triangle, where $\underline{f}' = \underline{m} \circ \underline{\ell}'$, and satisfy

$$\begin{aligned}
\underline{g}' \circ \underline{m} &= \underline{g} \circ \underline{m}' \quad , \quad \underline{q}' \circ \underline{g} = \underline{v}', \\
\underline{q} \circ \underline{g}' &= \underline{v} \quad , \quad S\underline{\ell}' \circ \underline{q}' + S\underline{\ell} \circ \underline{q} = 0.
\end{aligned}$$

$$\begin{array}{ccccccc}
X & \xrightarrow{\underline{f}} & Y & \xrightarrow{\underline{v}'} & SX' & \xrightarrow{S\underline{\ell}'} & SM \\
& \searrow \underline{\ell} & \nearrow \underline{m}' & \searrow \underline{g} & \nearrow \underline{q}' & \searrow & \\
& M & & Z & & & \\
& \nearrow \underline{\ell}' & \searrow \underline{m} & \nearrow \underline{g}' & \searrow \underline{q} & \nearrow & \\
X' & \xrightarrow{\underline{f}'} & Y' & \xrightarrow{\underline{v}} & SX & \xrightarrow{-S\underline{\ell}} & SM
\end{array}$$

We may assume (6.4), (6.5), (6.6) are standard triangles, arising from the following morphisms of conflations.

$$\begin{array}{ccccccc}
\Omega Y' \rightarrow X \xrightarrow{\underline{\ell}} M \xrightarrow{\underline{m}} Y' \xrightarrow{\underline{n}} \Sigma X & , & \Omega Y \rightarrow X' \xrightarrow{\underline{\ell}'} M \xrightarrow{\underline{m}'} Y \xrightarrow{\underline{n}'} \Sigma X' \\
\downarrow \circlearrowleft \parallel \circlearrowleft \downarrow u \circlearrowleft \downarrow v \circlearrowleft \parallel & , & \downarrow \circlearrowleft \parallel \circlearrowleft \downarrow u' \circlearrowleft \downarrow v' \circlearrowleft \parallel \\
\Omega S_X \rightarrow X \xrightarrow{\alpha_X} I_X \xrightarrow{\beta_X} S_X \xrightarrow{\gamma_X} \Sigma X & , & \Omega S_{X'} \rightarrow X' \xrightarrow{\alpha_{X'}} I_{X'} \xrightarrow{\beta_{X'}} S_{X'} \xrightarrow{\gamma_{X'}} \Sigma X'
\end{array}$$

$$(6.7) \quad \begin{array}{ccccccc} \Omega Z & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \downarrow & \circlearrowleft & \parallel & \circlearrowleft & \downarrow p & \circlearrowleft & \downarrow q \circlearrowleft \parallel \\ \Omega S_X & \longrightarrow & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X \xrightarrow{\gamma_X} \Sigma X \end{array}$$

Claim 6.15. *We may assume $m' \circ \ell = f$.*

Proof of Claim 6.15. Since $\underline{m'} \circ \underline{\ell} = \underline{f}$, there exist $I \in \mathcal{I}_D$, $f_1 \in \mathcal{Z}(X, I)$ and $f_2 \in \mathcal{Z}(I, Y)$ such that $f_2 \circ f_1 = f - m' \circ \ell$. Let $i_M: M \rightarrow M \oplus I$ and $p_M: M \oplus I \rightarrow M$ be the inclusion and the projection, respectively. By Corollary 4.3 and Lemma 4.4, we have extensions

$$\begin{aligned} \Omega Q &\rightarrow X \xrightarrow{(\ell, f_1)} M \oplus I \rightarrow Q \rightarrow \Sigma X, \\ \Omega M &\rightarrow I \rightarrow M \oplus I \xrightarrow{p_M} M \rightarrow \Sigma I. \end{aligned}$$

By Proposition 4.7, we obtain the following morphisms of extensions by Lemma 6.12.

$$\begin{array}{ccccccc} & & & \Sigma I & & & \\ & & & \uparrow & \swarrow & & \\ & & & M & \xrightarrow{m} & Y' & \\ & & \ell \nearrow & \uparrow p_M & \circlearrowleft \exists \rho & \downarrow n & \\ \Omega Q & \longrightarrow & X & \longrightarrow & M \oplus I & \longrightarrow & Q \longrightarrow \Sigma X \\ & & \uparrow & \searrow (\ell, f_1) & \uparrow & & \\ & & \Omega Y' & \dashrightarrow & I & & \\ & & & \uparrow & & & \\ & & & \Omega M & & & \end{array}$$

Thus we have $Q \in \mathcal{Z}$, and obtain an isomorphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{(\ell, f_1)} & M \oplus I & \longrightarrow & Q & \longrightarrow & SX \\ \parallel & \circlearrowleft & \cong \downarrow p_M & \circlearrowleft & \cong \downarrow \rho & \circlearrowleft & \parallel \\ X & \xrightarrow{\underline{\ell}} & M & \xrightarrow{\underline{m}} & Y' & \longrightarrow & SX \end{array}$$

Dually, there exist morphisms of extensions

$$\begin{array}{ccccccc} & & & \Omega I & & & \\ & & & \downarrow & \searrow & & \\ & & & M & \xleftarrow{\ell'} & X' & \\ & & m' \swarrow & \downarrow i_M & \circlearrowleft \exists \omega & \downarrow & \\ \Sigma R & \longleftarrow & Y & \longleftarrow & M \oplus I & \longleftarrow & \exists R \longleftarrow \Omega X' \\ & & \downarrow n' & \downarrow m' + f_2 & \downarrow & & \\ & & \Sigma X' & \dashleftarrow & I & & \\ & & & \downarrow & & & \\ & & & \Sigma M & & & \end{array}$$

which implies $R \in \mathcal{Z}$ and yields an isomorphism of distinguished triangles

$$\begin{array}{ccccccc} X' & \xrightarrow{\ell'} & M & \xrightarrow{m'} & Y & \longrightarrow & SX' \\ \cong \downarrow \omega & \circ & \cong \downarrow i_M & \circ & \parallel & \circ & \cong \downarrow S\omega \\ R & \longrightarrow & M \oplus I & \xrightarrow{m'+f_2} & Y & \longrightarrow & SR \end{array}$$

Thus, replacing ℓ by (ℓ, f_1) and m' by $m' + f_2$, we may assume $m' \circ \ell = f$. \square

By Claim 6.15, assume $m' \circ \ell = f$. Then by Proposition 4.7, there exist $g' \in \mathcal{Z}(Y', Z)$ and $h' \in \mathcal{C}(Z, \Sigma X')$ such that

$$\Omega Z \rightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{h'} \Sigma X'$$

is a conflation, and make the following diagram commutative.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{n'} & \Sigma X' & & \\ \ell \searrow & \circ & \nearrow m' & g \searrow & \nearrow h' & \circ & \searrow \Sigma \ell' \\ & M & & Z & & \Sigma M \\ \ell' \nearrow & \circ & \searrow m & g' \nearrow & \searrow h & \circ & \nearrow -\Sigma \ell \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{n} & \Sigma X & & \end{array}$$

If we take a morphism of conflations

$$\begin{array}{ccccccc} \Omega Z & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z \xrightarrow{h'} \Sigma X' \\ \downarrow & \circ & \parallel & \circ & \downarrow p' & \circ & \downarrow q' \circ \parallel \\ \Omega S_{X'} & \longrightarrow & X' & \xrightarrow{\alpha_{X'}} & I_{X'} & \xrightarrow{\beta_{X'}} & S_{X'} \xrightarrow{\gamma_{X'}} \Sigma X' \end{array}$$

then by Lemma 6.12, we obtain morphisms of standard triangles

$$\begin{array}{ccccccc} X' & \xrightarrow{\ell'} & M & \xrightarrow{m'} & Y & \xrightarrow{v'} & SX' \\ \parallel & \circ & \downarrow \underline{m} & \circ & \downarrow \underline{g} & \circ & \parallel \\ X' & \xrightarrow{\underline{f}'} & Y' & \xrightarrow{\underline{g}'} & Z & \xrightarrow{\underline{q}'} & SX' \end{array} \quad \text{and} \quad \begin{array}{ccccccc} X & \xrightarrow{\ell} & M & \xrightarrow{m} & Y' & \xrightarrow{v} & SX \\ \parallel & \circ & \downarrow \underline{m'} & \circ & \downarrow \underline{g'} & \circ & \parallel \\ X & \xrightarrow{\underline{f}} & Y & \xrightarrow{\underline{g}} & Z & \xrightarrow{\underline{q}} & SX \end{array}$$

Thus it remains to show $S\underline{\ell'} \circ \underline{q'} + S\underline{\ell} \circ \underline{q} = 0$.

Claim 6.16. *There exist morphisms of conflations*

$$(6.8) \quad \begin{array}{ccccccc} \Omega Z & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \\ \downarrow & \circ & \downarrow \ell & \circ & \downarrow r & \circ & \downarrow s \circ \downarrow \Sigma \ell \\ \Omega S_M & \longrightarrow & M & \xrightarrow{\alpha_M} & I_M & \xrightarrow{\beta_M} & S_M \xrightarrow{\gamma_M} \Sigma M \end{array}$$

$$(6.9) \quad \begin{array}{ccccccc} \Omega Z & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z \xrightarrow{h'} \Sigma X' \\ \downarrow & \circ & \downarrow \ell' & \circ & \downarrow r' & \circ & \downarrow s' \circ \downarrow \Sigma \ell' \\ \Omega S_M & \longrightarrow & M & \xrightarrow{\alpha_M} & I_M & \xrightarrow{\beta_M} & S_M \xrightarrow{\gamma_M} \Sigma M \end{array}$$

such that

$$r \circ m' + r' \circ m = \alpha_M.$$

Moreover, s and s' satisfy

$$(6.10) \quad \underline{s} = S\underline{\ell} \circ \underline{q} \quad \text{and} \quad \underline{s}' = S\underline{\ell}' \circ \underline{q}'.$$

Suppose Claim 6.16 is shown. Then by

$$\begin{aligned} (s + s') \circ g \circ m' &= s \circ g \circ m' + s' \circ g' \circ m \\ &= \beta_M \circ r \circ m' + \beta_M \circ r' \circ m \\ &= \beta_M \circ \alpha_M = 0, \end{aligned}$$

there exists $w' \in \mathcal{C}(\Sigma X', S_M)$ such that $w' \circ n' = (s + s') \circ g$. Thus by $((s + s') - w' \circ h') \circ g = 0$, there exists $w \in \mathcal{C}(\Sigma X, S_M)$ such that $w \circ h = s + s' - w' \circ h'$, namely

$$s + s' = w \circ h + w' \circ h'.$$

Take a conflation

$$\Omega Z \rightarrow X_0 \rightarrow I_0 \xrightarrow{\beta_0} Z \xrightarrow{\gamma_0} \Sigma X_0$$

with $I_0 \in \mathcal{I}_{\mathcal{D}}$. We have morphisms of conflations

$$\begin{array}{ccccccc} \Omega Z & \rightarrow & X_0 & \rightarrow & I_0 & \xrightarrow{\beta_0} & Z \xrightarrow{\gamma_0} \Sigma X_0 \\ \parallel & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\ \Omega Z & \rightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \xrightarrow{h} \Sigma X \end{array} \quad , \quad \begin{array}{ccccccc} \Omega Z & \rightarrow & X_0 & \rightarrow & I_0 & \xrightarrow{\beta_0} & Z \xrightarrow{\gamma_0} \Sigma X_0 \\ \parallel & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\ \Omega Z & \rightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z \xrightarrow{h'} \Sigma X' \end{array}$$

and thus obtain

$$s + s' = (w \circ \xi + w' \circ \xi') \circ \gamma_0.$$

Since $\gamma_M \circ (s + s') = (\Sigma \ell) \circ h + (\Sigma \ell') \circ h' = 0$, we can conclude that $s + s'$ factors through I_M by (AC1).

$$\begin{array}{ccccccc} \Omega Z & \rightarrow & X_0 & \rightarrow & I_0 & \xrightarrow{\beta_0} & Z \xrightarrow{\gamma_0} \Sigma X_0 \\ & & & & \searrow & \downarrow s+s' & \\ \Omega S_M & \rightarrow & M & \xrightarrow{\alpha_M} & I_M & \xrightarrow{\beta_M} & S_M \xrightarrow{\gamma_M} \Sigma M \end{array}$$

By (6.10), this means $S\underline{\ell}' \circ \underline{q}' + S\underline{\ell} \circ \underline{q} = 0$, and Proposition 6.14 can be shown. Thus it suffices to show Claim 6.16.

Proof of Claim 6.16. By $I_M \in \mathcal{I}_{\mathcal{D}}$, there exists $r \in \mathcal{Z}(Y, I_M)$ such that $r \circ f = \alpha_M \circ \ell$. By $(\alpha_M - r \circ m') \circ \ell = 0$, there exists $r' \in \mathcal{Z}(Y', I_M)$ such that $r' \circ m = \alpha_M - r \circ m'$. By (RTR3), there exist $s, s' \in \mathcal{Z}(Z, S_M)$ such that (6.8) and (6.9) are morphisms of conflations.

By definition, S_ℓ is a morphism which gives a morphism of conflations as follows.

$$\begin{array}{ccccccc} \Omega S_X & \rightarrow & X & \xrightarrow{\alpha_X} & I_X & \xrightarrow{\beta_X} & S_X \xrightarrow{\gamma_X} \Sigma X \\ \downarrow & \circlearrowleft & \downarrow \ell & \circlearrowleft & \downarrow I_\ell & \circlearrowleft & \downarrow S_\ell \circlearrowleft \downarrow \Sigma \ell \\ \Omega S_M & \rightarrow & M & \xrightarrow{\alpha_M} & I_M & \xrightarrow{\beta_M} & S_M \xrightarrow{\gamma_M} \Sigma M \end{array}$$

Composing with (6.7), we obtain a morphism of conflations

$$\begin{array}{ccccccccc}
 \Omega Z & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
 \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow I_\ell \circ p & \circlearrowleft & \downarrow S_\ell \circ q & \circlearrowleft & \downarrow \Sigma \ell \\
 \Omega S_M & \longrightarrow & M & \xrightarrow{\alpha_M} & I_M & \xrightarrow{\beta_M} & S_M & \xrightarrow{\gamma_M} & \Sigma M
 \end{array}$$

Thus, comparing with (6.8), we obtain $\underline{s} = S\underline{\ell} \circ \underline{q}$ by Lemma 6.3. Similarly for s' . \square

\square

By the above arguments, we obtain the following.

Theorem 6.17. *Let \mathcal{C} be a pseudo-triangulated category satisfying Condition 6.1, and let $\mathcal{Z} \subseteq \mathcal{C}$ be an extension-closed subcategory, and let $\mathcal{D} \subseteq \mathcal{Z}$ is a full additive replete subcategory closed under finite direct summands in \mathcal{Z} . If $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius, then $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$ becomes a triangulated category.*

In particular, if \mathcal{Z} is Frobenius, then the stable category \mathcal{Z}/\mathcal{I} becomes a triangulated category.

7. POSSIBILITY OF FURTHER GENERALIZATIONS

In [B], for any triangulated category \mathcal{C} , Beligiannis showed that if we are given a proper class of triangles \mathcal{E} on \mathcal{C} satisfying some conditions similar to the Frobenius condition discussed in section 5, then $\mathcal{C}/\mathcal{P}(\mathcal{E})$ becomes triangulated (Theorem 7.2 in [B]). Here, $\mathcal{P}(\mathcal{E})$ is the subcategory of ‘projectives’, defined in a similar, but different manner (Definition 4.1 in [B]). With that definition, $\mathcal{P}(\mathcal{E})$ becomes closed under Σ , but this conflicts with Iyama-Yoshino’s construction, in which the factoring category \mathcal{D} satisfies $\mathcal{C}(\mathcal{D}, \Sigma \mathcal{D}) = 0$. We wonder if there exists a general construction unifying the construction in [B] and that in section 6.

We also remark that there is another very general construction of a triangulated stable category. In [BM], Beligiannis and Marmaridis constructed a left triangulated category (in the sense of [B] or [BM]) from a pair $(\mathcal{C}, \mathcal{X})$ of an additive category \mathcal{C} and a contravariantly finite subcategory \mathcal{X} assuming some existence condition on kernels (Theorem 2.12 in [BM]). Therefore if \mathcal{X} is functorially finite and satisfies some nice properties, it is expected that this resulting category becomes triangulated. In fact, Happel’s construction is one of these cases (Remark 2.14 in [BM]). Although this existence condition is not satisfied by a triangulated category \mathcal{C} unless we replace it by some ‘pseudo’ one, we hope some unifying construction will be possible.

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